

MTH250 - CALCULUS

Lecture No. 27

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These lecture notes are prepared to serve as handouts for the video lectures on the course *MTH-250 Calculus* of Virtual Campus, COMSATS Institute of Information Technology, Islamabad, Pakistan. The notes were derived from an extensive collection of notes and books. Most of the theoretical details as well as worked problems are from *Calculus*, by *H. Anton, I Bivens and S. Davis*, John Wiley & Sons, 10th edition, 2012.

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LECTURE 27

Double integrals in polar coordinates

Some double integrals are easier to evaluate if the region of integration is expressed in polar coordinates. This is usually true if the region is bounded by a curve whose equation is simpler in polar coordinates than in rectangular coordinates.

27.1 Double integrals in polar coordinates

Definition 27.1 (Simple polar region).

A simple polar region in a polar coordinate system is a region that is enclosed between two rays, $\theta = \alpha$ and $\theta = \beta$, and two continuous polar curves, $r = r_1(\theta)$ and $r = r_2(\theta)$, where the equations of the rays and the polar curves satisfy the following conditions:

1. $\alpha \leq \beta$
2. $\beta - \alpha < 2\pi$
3. $0 \leq r_1(\theta) \leq r_2(\theta)$.

Definition 27.2 (Polar rectangle).

A polar rectangle is a simple polar region for which the bounding polar curves are cir-

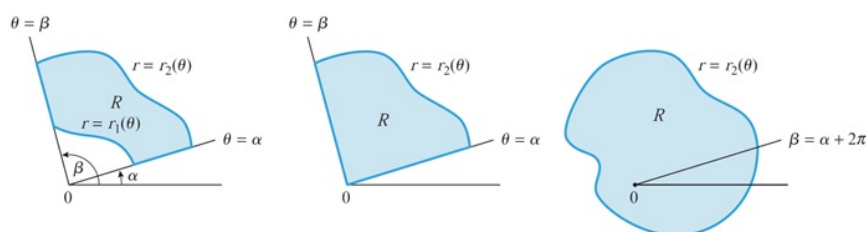
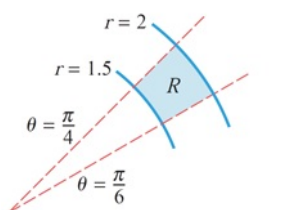


Figure 27.1: The simple polar region.

cular arcs. For example, the polar rectangle R given by $1.5 \leq r \leq 2$, $\pi/6 \leq \theta \leq \pi/4$ is polar rectangle; see Figure 27.2.

Figure 27.2: The polar rectangle $1.5 \leq r \leq 2$, $\pi/6 \leq \theta \leq \pi/4$.

Definition 27.3 (the volume problem in polar coordinates).

Given a function $f(r, \theta)$ that is continuous and non-negative on a simple polar region R , find the volume of the solid that is enclosed between the region R and the surface whose equation in cylindrical coordinates is $z = f(r, \theta)$.

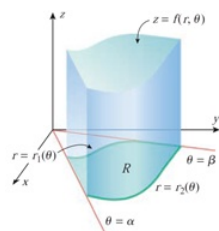
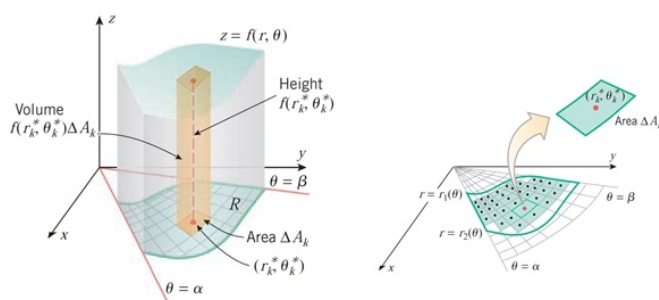


Figure 27.3: The volume problem

In order to address the volume problem, we proceed as follows.

1. Using circular arcs and rays, divide the simple polar region R into n polar-rectangles.
2. Choose any arbitrary point (r_k^*, θ_k^*) in each sub-rectangle.

Figure 27.4: Dividing the polar region R into n polar rectangles.

3. Let ΔA_k denote the area of the k^{th} rectangle.
4. The volume of a polar rectangular parallelepiped with height $f(r_k^*, \theta_k^*)$ and base area ΔA_k is given by $f(r_k^*, \theta_k^*) \Delta A_k$.
5. Approximation to the volume V of the entire solid is

$$R_n = \sum_{k=1}^n f(r_k^*, \theta_k^*) \Delta A_k.$$

The sum R_n is called the n^{th} polar Riemann sum.

Definition 27.4 (Volume under a surface).

If $f(r, \theta)$ is continuous function and has both positive and negative values, then the limit region R in the $r\theta$ -plane, then the volume of the solid enclosed between the surface $z = f(r, \theta)$ and the region R is defined by

$$V = \lim_{n \rightarrow +\infty} \sum_{k=1}^n f(r_k^*, \theta_k^*) \Delta A_k$$

represents the net signed volume between the region R and the surface $z = f(r, \theta)$ (as with the double integrals in rectangular coordinates).

Definition 27.5 (Polar double integral).

The double integral of the function $f(r, \theta)$ over a region R is defined as the limit of the Riemann sums and is denoted by

$$\iint_R f(r, \theta) dA = \lim_{n \rightarrow +\infty} \sum_{k=1}^n f(r_k^*, \theta_k^*) \Delta A_k.$$

If f is continuous and non-negative on the region R , then the volume V can be related to the double integral as

$$V = \iint_R f(r, \theta) dA.$$

Remark 27.6. As

$$R_k = \{(r, \theta) \mid r_{k-1} \leq r \leq r_k, \quad \theta_{k-1} \leq \theta \leq \theta_k\},$$

the area ΔA_k of R_k is given by

$$\Delta A_k = \frac{1}{2} r_k^2 \Delta \theta_k - \frac{1}{2} r_{k-1}^2 \Delta \theta_{k-1} = \frac{1}{2} (r_k + r_{k-1})(r_k - r_{k-1}) \Delta \theta_k = r_k^* \Delta r_k \Delta \theta_k$$

where $r_k^* = \frac{1}{2}(r_k + r_{k-1})$. Therefore,

$$f(r_k^*, \theta_k^*) \Delta A_k = f(r_k^*, \theta_k^*) r_k^* \Delta r_k \Delta \theta_k.$$

Theorem 27.7. If R is a simple polar region whose boundaries are the rays $\theta = \alpha$ and $\theta = \beta$ and the curves $r = r_1(\theta)$ and $r = r_2(\theta)$ and if $f(r, \theta)$ is continuous on R , then

$$\iint_R f(r, \theta) dA = \int_{\alpha}^{\beta} \int_{r_1(\theta)}^{r_2(\theta)} f(r, \theta) dr d\theta.$$

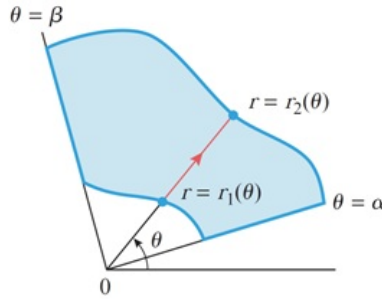


Figure 27.5: Polar double integral boundaries are the rays $\theta = \alpha$ and $\theta = \beta$ and the curves $r = r_1(\theta)$ and $r = r_2(\theta)$.

27.1.1 Determining limits of integration for a polar double integral

In order to find the limits of integration for a polar double integral over a simple polar region, we have follow the procedure given below

1. Since θ is held fixed for the first integration, draw a radial line from the origin through the region R at a fixed angle θ ; see Figure 27.6 This line crosses the boundary of R at most twice. The innermost point of intersection is on the inner boundary curve $r = r_1(\theta)$ and the outermost point is on the outer boundary curve $r = r_2(\theta)$. These intersections determine the r -limits of integration.

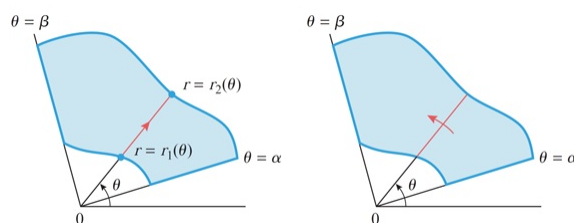
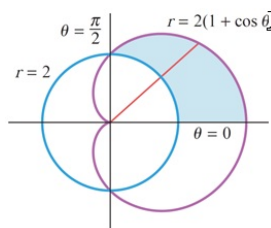


Figure 27.6: Finding the limits of integration.

2. Imagine rotating the radial line from *Step 1* about the origin, thus sweeping out the region R . The least angle at which the radial line intersects the region R is $\theta = \alpha$ and the greatest angle is $\theta = \beta$; see Figure 27.6. This determines the θ -limits of integration.

Example 27.1. Evaluate $\iint_R \sin \theta dA$, where R is the region in the first quadrant that is outside the circle $r = 2$ and inside the cardioid $r = 2(1 + \cos \theta)$.

Figure 27.7: Region in the first quadrant that is outside the circle $r = 2$ and inside the cardioid $r = 2(1 + \cos \theta)$.

Solution. Following the two steps outlined above, we obtain

$$\begin{aligned}
 \iint_R \sin \theta dA &= \int_0^{\pi/2} \int_2^{2(1+\cos \theta)} (\sin \theta) r dr d\theta \\
 &= \int_0^{\pi/2} \left[\frac{1}{2} r^2 \sin \theta \right]_{r=2}^{2(1+\cos \theta)} d\theta \\
 &= 2 \int_0^{\pi/2} [(1 + \cos \theta)^2 \sin \theta - \sin \theta] d\theta \\
 &= 2 \left[-\frac{1}{3} (1 + \cos \theta)^3 + \cos \theta \right]_0^{\pi/2} \\
 &= 2 \left[-\frac{1}{3} - \left(-\frac{5}{3} \right) \right] = \frac{8}{3}.
 \end{aligned}$$

□

27.2 Finding area using polar double integrals

Recall that the area of a region R in the xy -plane can be expressed as

$$\text{Area of } R = \iint_R 1 dA = \iint_R dA.$$

The argument used to derive this result can also be used to show that the formula applies to polar double integrals over regions in polar coordinates.

Example 27.2. Use a polar double integral to find the area enclosed by the three-petaled rose $r = \sin 3\theta$.

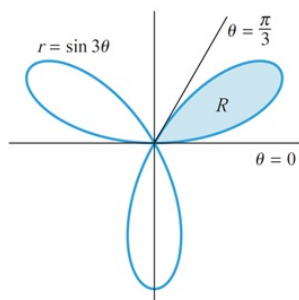


Figure 27.8: Three-petaled rose $r = \sin 3\theta$

Solution. The rose is sketched in Figure 27.8. We will calculate the area of the petal R in the first quadrant and multiply by three.

$$\begin{aligned} A &= 3 \iint_R dA = 3 \int_0^{\pi/3} \int_0^{\sin 3\theta} r dr d\theta. \\ &= \frac{3}{2} \int_0^{\pi/3} \sin^2 3\theta d\theta = \frac{3}{4} \int_0^{\pi/3} (1 - \cos 6\theta) d\theta \\ &= \frac{3}{4} \left[\theta - \frac{\sin 6\theta}{6} \right]_0^{\pi/3} = \frac{1}{4} \pi \end{aligned}$$

□

27.3 Changing coordinates in integrals

Sometimes a double integral that is difficult to evaluate in rectangular coordinates can be evaluated more easily in polar coordinates by making the substitution $x = r \cos \theta$, $y = r \sin \theta$ and expressing the region of integration in polar form; that is, we rewrite the double integral in rectangular coordinates as

$$\iint_R f(x, y) dA = \iint_R f(r \cos \theta, r \sin \theta) dA = \iint_{\text{appr. lim}} f(r \cos \theta, r \sin \theta) r dr d\theta.$$

Proposition 27.8.

1. Let $R = \{(r, \theta) \mid a \leq r \leq b, \alpha \leq \theta \leq \beta\}$ be a polar rectangle and $0 \leq \beta - \alpha \leq 2\pi$. If f is continuous on R , then

$$\iint_R f(x, y) dA = \int_{\alpha}^{\beta} \int_a^b f(r \cos \theta, r \sin \theta) r dr d\theta.$$

2. Let $D = \{(r, \theta) \mid \alpha \leq \theta \leq \beta, h_1(\theta) \leq r \leq h_2(\theta)\}$ be a polar region. If f is continuous on D , then

$$\iint_D f(x, y) dA = \int_{\alpha}^{\beta} \int_{h_1(\theta)}^{h_2(\theta)} f(r \cos \theta, r \sin \theta) r dr d\theta.$$

Example 27.3. Use polar coordinates to evaluate $\int_{-1}^1 \int_0^{\sqrt{1-x^2}} (x^2 + y^2)^{1/2} dy dx$.

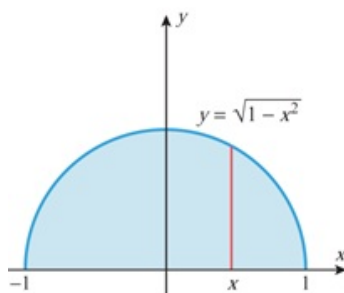


Figure 27.9: Semicircular region R .

Solution. In this problem we are starting with an iterated integral in rectangular coordinates rather than a double integral, so before we can make the conversion to polar coordinates we will have to identify the region of integration. To do this, we observe that for fixed x the y -integration runs from $y = 0$ to $y = \sqrt{1 - x^2}$, which tells us that the lower boundary of the region is the x -axis and the upper boundary is a semicircle of radius 1 centered at the origin. From the x -integration we see that x varies from -1 to 1 , so we conclude that the region of integration is as shown in Figure 27.9. In polar coordinates, this is the region swept out as r varies between 0 and 1 and θ varies between 0 and π . Thus,

$$\int_{-1}^1 \int_0^{\sqrt{1-x^2}} (x^2 + y^2)^{1/2} dy dx = \iint_R (x^2 + y^2)^{1/2} dA = \int_0^{\pi} \int_0^1 r^3 r dr d\theta = \int_0^{\pi} \frac{1}{5} d\theta = \frac{\pi}{5}.$$

□

Example 27.4. Find the volume of the solid bounded by the plane $z = 0$ and the paraboloid $z = 1 - x^2 - y^2$.

Solution. Here the region of integration is

$$D = \{(r, \theta) \mid 0 \leq r \leq 1, \quad 0 \leq \theta \leq 2\pi\}.$$

Therefore,

$$V = \iint_D (1 - x^2 - y^2) dA = \int_0^{2\pi} \int_0^1 (1 - r^2) r dr d\theta = \frac{\pi}{2}.$$

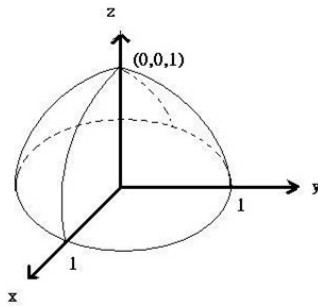


Figure 27.10: Region of integration D

□